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Mean-Extended Gini Portfolios: A 3D Efficient Frontier

Frank Hespeler[2] · **Haim Shalit**[1]

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Abstract Using a numerical optimization technique we construct the mean-extended Gini (MEG) efficient frontier as a workable alternative to the mean-variance efficient frontier. MEG enables the introduction of specific risk aversion into portfolio selection. The resulting portfolios are stochastically dominant for all risk-averse investors. Solving for MEG portfolios allows investors to tailor portfolios for specific risk aversion. The extended Gini is calculated by the covariance of asset returns with a weighing function of the cumulative distribution function (CDF) of these returns. In a sample of asset returns, the CDF is estimated by ranking returns. In this case, analytical optimization techniques using continuous gradient approaches are unavailable, thus the need to develop numerical optimization techniques. In this paper we develop a numerical optimization algorithm that finds the portfolio optimal frontier for arbitrarily large sets of shares. The result is a 3-dimension MEG efficient frontier in the space formed by mean, the extended Gini, and the risk aversion coefficient.

Keywords Mean-Gini portfolios · Numerical optimization · Stochastic dominance portfolios · 3D efficient frontier

1 Introduction

The mean-extended Gini (MEG) investment model offers an alternative to the standard mean-variance (MV) model by measuring risk using Gini's mean difference instead of

✉ Haim Shalit
shalit@bgu.ac.il

Frank Hespeler
frank.hespeler@sciencespo.fr

1 Department of Economics, Ben-Gurion University of the Negev, Beersheba, Israel

2 Sciences Po, Paris, France

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the standard deviation. MEG was developed by Yitzhaki (1983) to allow for the specific introduction of risk aversion differentiation into the risk decision process.The MEG approach was first used in finance by Shalit and Yitzhaki (1984) to price risky assets and construct efficient portfolios which are second-degree stochastic dominant (SSD). Later, Shalit and Yitzhaki (2005) provided superior alternative optimal allocations to the MV efficient frontier, in particular when risky assets are not normally distributed. We present a numerical optimization algorithm in Mathematica in order to construct mean-extended Gini efficient portfolios for large sets of assets with and without short-sales positions. The purpose is to familiarize the investment practitioner with the MEG model as a substitute to the MV model in portfolio selection. Solving for MEG portfolios enables investors to construct efficient portfolios that are tailored to their specific risk requirements. Indeed, when investors desire to hold riskier or less risky assets, MEG has the advantage of incorporating individual risk aversion. However, because the extended Gini is calculated by weighing the ranking function as a proxy for the cumulative distribution, analytical optimization techniques are unavailable. Hence, there is a need to develop numerical optimization techniques that would make MEG a superior tool for choosing optimal portfolios.

2 The Mean-Extended Gini Investment Model

We present a portfolio investment model in which investors minimize their risk as measured by the extended Gini, subject to given expected returns. The Gini index is a statistic of dispersion used mainly in income distribution studies in order to compute income inequality. In financial economics, it is customary to use as measure of risk the absolute value of the Gini index, called here the Gini. Its main feature is that it estimates the pure risk of asset x since it can be obtained from its generalized (absolute) Lorenz curve. To see this, assume that asset x's returns are distributed by the cumulative probability distribution function (CDF), $F(x)$. Following Gastwirth (1971), the absolute Lorenz curve is defined over the probability φ as:

$$L_x(\varphi) = \int_0^{\varphi} F^{-1}(t)dt \tag{1}$$

where $F^{-1}(t) = Inf\ \{x \mid F(x) \geqslant t\}$ is the inverse of the CDF. The absolute Lorenz curve of asset x starts at the origin ($\varphi = 0$) and ends up at the mean $E(x)$ for $\varphi = 1$. For identical means, the more convex the curve is, the riskier is the asset. The absolute Lorenz curve for the safest asset given a mean return is expressed by a straight line, called the line of safe asset (LSA), that runs from the origin to the mean of asset x. The Gini is calculated by the area between the LSA that yields the same mean return and its absolute Lorenz curve. It is the pure risk of the asset since, for every probability φ, had one invest in the risky asset, one would get the cumulative expected return along the absolute Lorenz curve whereas investing in the riskless asset one would obtain a higher cumulative expected return on the LSA. Therefore, the farther the LSA is from the absolute Lorenz curve, the greater is the risk assumed by the asset.

In financial applications, the Gini is more conveniently expressed as $\Gamma_x = 2\text{cov}[x, F(x)]$; i.e., the covariance between asset returns x and their cumulative probability distribution $F(x)$. The latter is estimated by i/T, which is the relative ranking of x_i sorted from the lowest return ($i = 1$) to the highest ($i = T$).

As a measure to value uncertainty in risky assets, the Gini has been shown by Yitzhaki (1982) to have many advantages over the variance. Firstly, when returns depart from normality the Gini exhibits a better picture of the dispersion of the distribution since it compares the spread of observations among themselves. Secondly, together with the expected return, the Gini provides necessary and sufficient conditions for second degree stochastic dominance (SSD), implying that the mean and the Gini are a two parameter approach that is fully compatible with expected utility maximization. This feature is nonexistent with the variance and the mean unless asset returns are normally distributed or investor's preferences are quadratic.

An additional advantage of using the Gini as measure of risk is that it can be extended into a family of increasingly risk-averse models as developed by Yitzhaki (1983). Indeed, by adding one extra parameter, $\nu > 1$, the extended Gini measures asset risk when the lower portions of the returns distribution are multiplied by larger relative weights that express the concern investors have for losses when investing in risky assets. To specify increasing risk aversion, we derive the extended-Gini statistic by stressing the lower segments of the distribution of asset returns. The simple Gini is obtained by calculating the area between the LSA and the Lorenz curve. Similarly, we obtain the extended Gini by adding the relatively weighted vertical differences between the LSA and the Lorenz curve. This area is calculated using the parameter ν to obtain the extended Gini of asset x as follows:

$$\Gamma_x(\nu) = \nu(\nu - 1) \int_0^1 (1 - \varphi)^{\nu-2} \left[\varphi E(x) - L_x(\varphi)\right] d\varphi \tag{2}$$

where $L_x(\varphi)$ is the Lorenz curve from Eq. (1), $\varphi E(x)$ is the LSA, and $\nu(\nu - 1)(1 - \varphi)^{\nu-2}$ are the weights associated with each portion of the area between the LSA and the Lorenz curve. The parameter $\nu(> 0)$ is the risk aversion coefficient chosen by analysts to represent the relative fear of losses by investors. Some special cases of interest for the extended Gini parameter include the following: For $\nu = 2$ Eq. (2) becomes the simple Gini. For $\nu \to \infty$ the extended Gini reflects the attitude of a max-min investor who expresses risk only in terms of the worst outcome. For $\nu \to 1$, Eq. (2) cancels out, allowing risk-neutral investors without measures of dispersion to evaluate risk. For $0 < \nu < 1$ the extended Gini is negative and relates to risk-loving investors. For ease of presentation and because we are dealing with risk-averse investors, we consider here only the extended Gini with $\nu > 1$, although many of the results can be applied without modification to risk-loving investors. In financial analysis, it is easier to express the extended Gini using the covariance formula rather than Eq. (2):

$$\Gamma_x(\nu) = -\nu \mathrm{cov} \left\{x, [1 - F(x)]^{\nu-1}\right\} \tag{3}$$

To understand the essence of risk aversion using the extended Gini, the reader is referred to the swimmer/shark metaphor from Shalit and Yitzhaki (2009, p. 761): "…. As an example, imagine a shark is roaming the coastal waters. A risk-neutral swimmer will calculate the swimming benefits by using the objective probability of being struck by a shark. If the swimmer uses $\nu = 2$, she will attach as the probability of being struck, twice her entrance into the water although she will jump only once. If the swimmer uses $\nu \to \infty$, although she intends to enter the water only once, her behavior is as if she will be entering an infinite number of times. That is, if there is a tiny objective probability of having a shark roaming the waters, the behavior of the $\nu \to \infty$ swimmer is as if the shark will strike with a probability of one ...." Hence, we can see that the parameterν and the extended Gini span an entire continuous spectrum of risk-aversion behavior.

From an investor's point of view, given a specific ν, efficient portfolio frontiers can be constructed with and without allowing for short sales. This permits investors to construct efficient portfolios that are tailored to their specific risk needs. Indeed, when investors desire to hold riskier or less risky assets, MEG has the advantage of incorporating individual risk-aversion in the choice process itself without relying on the portfolio separation theorem. The investor's problem is to choose the positions that minimize the extended Gini of a portfolio of assets subject to a given mean as follows:

Consider a portfolio p of N assets whose weights are w_i and whose returns are given by $r_p = \sum_i^N w_i r_i$, where r_i are the assets' returns. A portfolio of assets requires that $\sum_i^N w_i = 1$. Hence,

$$\begin{aligned} \text{Minimize} \quad & -\nu \textstyle\sum_{i=1}^N w_i \text{cov}\left\{x_i, \left[1 - F_p(p)\right]^{\nu-1}\right\} \\ \text{subject to} \quad & E(p) = \textstyle\sum_{i=1}^N w_i E(x_i) \\ & 1 = \textstyle\sum_{i=1}^N w_i \end{aligned} \tag{4}$$

Changing the required mean allows the financial analyst to span the entire efficient frontier. The advantage of MEG is rooted in the different number of efficient frontiers each of which depends on the coefficient of risk aversion, ν (see Shalit and Yitzhaki 1989). Investors have the choice to opt for the portfolios that best suit their aversion to risk. Asset allocation using MEG is somewhat similar to MV portfolio optimization when short sales are allowed and when return distributions are exchangeable. In that case, standard MV algorithms can be used for MEG as done by Shalit and Yitzhaki (2005). Problem (4), although similar in structure to the MV optimization problem, is much more complex than the MV problem because the extended Gini of a portfolio cannot be derived as a simple function of the probability distribution statistics of the assets. Hence, a specific optimization programming is needed to solve the portfolio allocation problem whose technique is presented in the next section.

3 The Optimization Model

We construct MEG efficient portfolios by developing an algorithm based on numerical optimization. In practice, the Gini of a random variable can be either calculated by averaging the absolute differences between all observations pairs or by estimating the CDF by the rank function and applying it in the covariance formula. Because Gini derivatives are discontinuous, researchers are refrained from using analytical solutions to construct optimal MEG portfolios as gradient-type optimizations approaches fail.

Recently, a simple solution to the mean-Gini portfolio optimization problem was obtained by Cheung et al. (2005) using a standard Excel spreadsheet. That approach can be implemented to MEG for a smaller number of securities allowing for risk aversion differentiation. Our challenge was to use an advanced software package such as Mathematica to develop a reliable numerical optimization to find efficient portfolios that minimize the extended Gini subject to required expected returns for a large number of securities with and without short sales.

Visualized in Fig. 1, our algorithm is structured into a main routine that calls on subroutines responsible for minimizing the (extended) Gini or constructing efficient frontiers for a set of risk aversion parameters ν. After loading the input data, the main routine offers several parameter choices. The value for RestrictRange determines whether the (extended) Gini is minimized unconstrained (0) or subject to a set of required portfolio returns (1). In the latter case, the routine asks for the lower bound return (MPR), the step size (RPS) and the number of steps (NumberofSteps). It calculates the returns for which the efficient frontier is computed. The analyst is then asked to choose the risk measures by specifying values for MethodRange: 0 for the Gini, 1 for the extended Gini, 2 for both risk measures, or 3 for different extended Ginis. For the latter, the routine asks for the choice of ν or the information needed to construct a set of νs: the lower bound (RiskAversionStart), the step size (RiskAversionSize), and the number of steps (RiskAversionSteps). For MethodRange equals 3, a 3-D efficient frontier is spanned in the space expected return, extended Gini, and the risk-aversion $E(p)$, $\Gamma_p(\nu)$, ν. Finally, the analyst specifies whether short sales are allowed or not.

The program loads the two subroutines OptPortfolioGini and OptPortfolioExtGini that minimize the portfolio's (extended) Gini for a required return over eligible portfolio weights. These two routines call for the subroutine PortfolioGini to compute the Gini and for the subroutine PortfolioExtGini to compute the extended Gini. In both cases, a linear programming technique based on a simplex algorithm is used.

Conditional on the values for MethodRange and RestrictRange the main routine takes different paths. If the unconstrained (extended) Gini is chosen, the program executes OptPortfolioGini (OptPortfolioExtGini) and reports the results in the object ResultVector. If an efficient frontier is to be computed, the program enters alternative loops depending on the value chosen for MethodRange. If only one risk aversion coefficient is specified, for each step of the loop, starting from the minimum required return, OptPortfolioGini, OptPortfolioExtGini or both are executed, the results are added to ResultVector (ResultbVector or both of them) and the required return is increased by RPS.

Finally, if, according to MethodRange, a set of risk-aversion coefficients is chosen, the routine enters the alternative branch of two nested loops. The outer loop runs through the various risk aversions and the inner loop runs through the various required returns. On exiting the inner loop, the risk aversion is increased by RiskAversionSize and the results are added to ResultArray, then the routine proceeds to the next step of the outer loop. This procedure is repeated for RiskAversionSteps times.

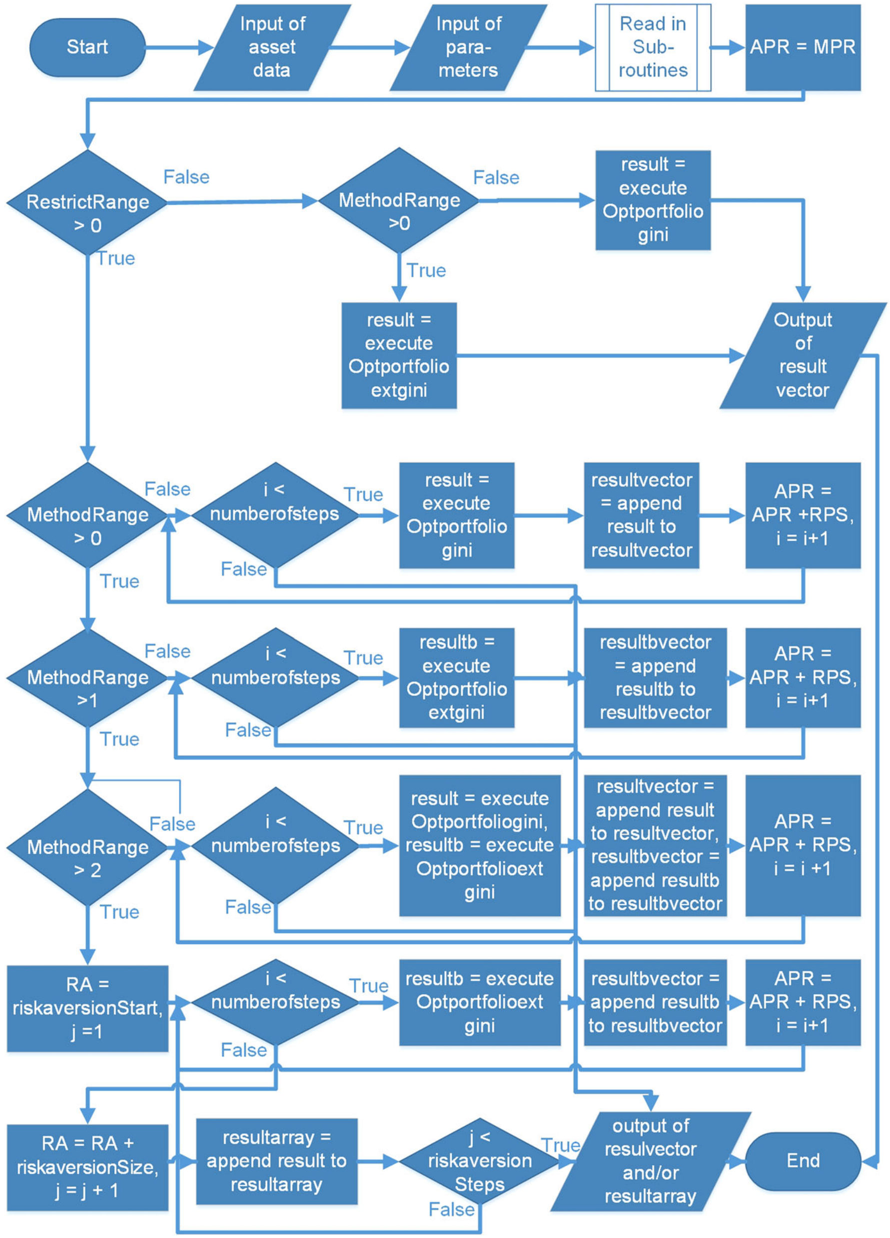

Fig. 1 Optimization algorithm flowchart

4 Empirical Analysis and Results

We use as data the monthly returns of 100 most valued traded stocks on the US financial markets from March 1992 until June 2007. Most of these firms appear in the S&P

100 index. The 183 returns were calculated from monthly close price adjusted for dividends and splits downloaded from finance.yahoo.com. The summary statistics (mean, standard deviation and Gini mean difference) are presented in Table 1 together with the Jarque-Bera test statistic for normality. For most of the firms normal distribution is rejected, justifying the use of the Gini as an appropriate risk measure to obtain SSD portfolios.

Using the Mathematica software, the optimization generates efficient portfolios frontiers by varying the mean return, the extended Gini, and the risk aversion parameter. The resulting 3-dimentional efficient frontier surface is displayed in Fig. 2. In here, a rising risk aversion parameter follows an increase in the risk compensation needed for a given expected return reduction. Furthermore, the figure illustrates that the speed of trade-off changes does not follow a monotone pattern, but rather appears as a volatile process. On the margin, it turns out that not even the change in the trade-off is a strictly monotone function of the risk aversion parameter. This rather unexpected result cannot only be explained by the non-continuous adjustments in the optimal weights of the remaining assets whenever security is added or removed from the optimal portfolio, but also from trade-offs inherent in the risk aversion parameter itself. This argument can be explained using the elasticity of the extended Gini with respect to the risk aversion parameter ν. From Eq. (2) we obtain the derivative of the extended Gini. Therefore, the elasticity w.r.t. ν is:

$$\frac{\partial \Gamma_x(\nu)}{\partial \nu}\frac{\nu}{\Gamma_x} = \frac{2\nu - 1}{\nu - 1} + \nu \int_0^1 ln(1-\varphi)d\varphi \qquad (5)$$

The ratio $\frac{2\nu-1}{\nu-1}$ and the second term of Eq. (5) create trade-offs that can lead to non-monotony. In particular, this ratio exhibits singularity when $\nu \to 1$ or $\nu \to 0$ implying that near these values the extended Gini elasticity is non-monotonous. This feature adds on top of the trade-off between risk-bearing and diversification that can raise the required mean return.

5 Conclusions

We have presented a new approach to construct MEG portfolios by inserting the coefficient of risk-aversion into the optimization program. Hence, the results show a three dimensional frontier where the risk-aversion coefficient is chosen to enhance the risk inherent in the portfolios. Not only the results deliver stochastic dominant portfolios but they allow the analyst to offer a variety of alternatives for risk-averse investors. In addition, the paper provides some innovations of a more technical nature.

These include a Mathematica algorithm consisting of several interdependent Mathematica packages and a notebook that allows for the efficient computation of hulls by varying portfolios for a predefined set of assets. The size of this set is only restricted by the computational resources available. In any case, these resources can be improved by using the inherent parallel computing capacities of Mathematica. In this way, the

Table 1 Monthly returns statistics 100 US stocks March 1992–June 2007

Firm	Mean (%)	SD (%)	Gini (%)	JB Stat	Firm	Mean (%)	SD (%)	Gini (%)	JB Stat
AA	1.38	9.30	5.25	131	HPQ	1.64	10.88	6.14	6.76
ABT	1.09	5.96	3.36	6.26	IFF	0.78	6.45	3.64	36.34
AAPL	2.31	14.75	8.32	8.67	IBM	1.40	9.13	5.15	15.33
AEP	0.86	6.07	3.43	9.82	INTC	2.23	12.07	6.81	7.71
AES	2.21	16.37	9.235	263	IP	0.61	7.68	4.33	21.22
AIG	1.37	6.46	3.64	22.71	JNJ	1.22	6.05	3.41	0.42
AMGN	1.56	9.91	5.59	55.14	JPM	1.50	8.95	5.05	32.40
AVP	1.75	9.22	5.20	292	KO	0.86	6.45	3.64	14.84
AXP	1.66	6.87	3.88	49.9	LTD	1.21	9.84	5.55	11.00
BA	1.26	7.82	4.41	37.76	MCD	1.23	6.82	3.85	10.67
BAC	1.34	6.95	3.92	28.43	MDT	1.64	7.02	3.96	5.75
BAX	1.20	7.51	4.24	73.44	MER	2.06	9.57	5.40	10.53
BHI	1.58	9.82	5.54	5.47	MMM	1.15	5.94	3.35	29.54
BMY	0.83	6.82	3.85	28.73	MO	1.46	8.33	4.70	35.37
BNI	1.38	6.87	3.87	2.58	MRK	0.94	7.86	4.44	1.51
BUD	1.02	4.86	2.74	0.23	MSFT	1.95	10.18	5.74	30.74
BDK	1.25	8.60	4.85	3.87	MAY	0.67	11.94	6.74	1247
BC	1.07	9.57	5.40	69.95	MEE	1.46	12.63	7.13	41.94
C	2.06	8.39	4.74		NSC	1.18	7.99	4.51	13.19
CAT	1.92	8.36	4.72	31.22	NSM	2.27	16.32	9.21	8.23
CCU	2.54	10.42	5.88	18.58	NT	1.30	19.55	11.03	953
CI	1.84	8.92	5.03	222	ORCL	3.20	14.23	8.03	77.35
CL	1.41	7.15	4.04	94.87	OMX	0.96	8.77	4.95	1.96
CMCSA	1.66	9.39	5.30	7.51	OXY	1.58	7.65	4.32	14.84
COP	1.51	6.84	3.86	8.4	PEP	1.1	6.17	3.48	74.3
CPB	0.97	6.54	3.69	2.8	PFE	1.21	6.82	3.85	1.57
CSC	1.41	10.07	5.58	134	PG	1.22	6.17	3.48	389
CSCO	2.89	12.11	6.83	5.12	RF	0.91	5.72	3.23	9.66
CVS	1.30	7.84	4.42	28.54	ROK	2.36	9.30	5.24	83.38
CVX	1.35	5.56	3.14	14.86	RTN	0.92	8.48	4.78	162
CEN	1.52	8.76	4.94	5.47	RSH	1.56	11.38	6.42	0.98
DD	0.89	6.71	3.79	0.51	S	1.33	9.42	5.31	65.71
DELL	3.42	14.79	8.34	5.98	SLB	0.74	7.05	3.98	37.35
DIS	0.89	7.50	4.23	13.08	SO	1.58	5.42	3.06	27.79
DOW	1.03	7.59	4.28	131	T	1.12	7.31	4.12	11.74
EK	0.57	8.45	4.77	42.66	TEK	1.91	12.69	7.16	54.91
EMC	3.40	14.79	8.35	1.83	TGT	1.79	7.93	4.48	2.58
EP	1.28	10.96	6.18	150	TWX	4.03	16.08	9.07	88.95
ETR	1.65	6.42	3.62	23.56	TXN	2.50	12.89	7.27	10.17
EXC	1.55	6.85	3.86	50.88	TYC	1.66	9.86	5.56	207

Table 1 continued

Firm	Mean	Std Dev	Gini	JB Stat	Firm	Mean	Std Dev	Gini	JB Stat
F	0.86	9.56	5.39	20.96	USB	1.42	7.43	4.19	131
FDX	1.59	8.45	4.77	21.41	UTX	1.73	7.04	3.97	125
GD	2.70	9.10	5.1	1951	VZ	0.98	7.29	4.11	152
GE	1.36	6.08	3.43	3.49	WB	0.94	7.24	4.08	47.55
GM	0.88	9.61	5.42	0.49	WFC	1.56	6.59	3.72	19.20
HAL	1.87	10.97	6.19	9.69	WMB	2.18	13.01	7.34	267
HD	1.30	8.07	4.55	4.13	WMT	1.02	7.14	4.03	4.43
HET	2.225	10.74	6.06	112	WY	1.14	7.41	4.18	0.49
HNZ	0.93	5.58	3.15	0.79	XRX	1.18	12.79	7.21	505
HON	1.40	9.16	5.17	339	XOM	1.35	4.74	2.68	56.74

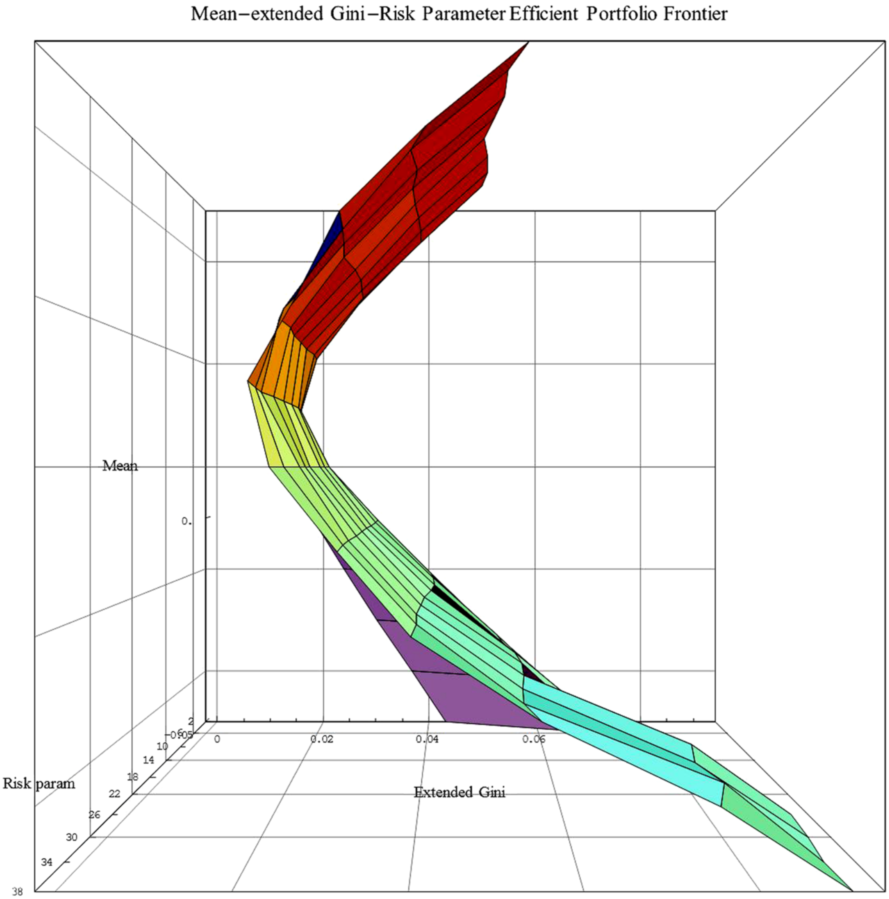

Fig. 2 3D efficient portfolios frontier

number of elements within the set of assets serving as the main input to the algorithm can be almost unlimited.

It is exactly this computational power that allows also for a high degree of flexibility in the design of the objective function within the process of the portfolio optimization. Hence, future extensions of this research could include taking into account higher moments and/or co-moments, additional parameters besides the risk-aversion parameter and even a more general functional form. Another potentially interesting extension would be to include an objective function that not only reflects the trade-off between return and risk, but also uses a measure of financial stability so that minor shocks on the exogenous parameters and variables would not result in major portfolio restructuring. Building on this argument, integrating transaction costs into the objective function could potentially convey interesting results as well.

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